Optimization modeling for groundwater management in the Central Sands of Wisconsin

Jesse Holzer

University of Wisconsin - Madison

Midwest Groundwater Conference, October 1, 2012

The problem of groundwater in the Central Sands

At least two highly antagonistic interests in groundwater use in the Central Sands.
1. Agriculture needs to pump groundwater to grow vegetables in sandy soil. This interest group is highly focused.
2. Very broad group of interests in maintaining and increasing lake levels and stream flows. Lake property owners, outdoors sportspeople.
Groundwater pumping and lake and stream levels are linked through the hydrological system, so any gain for lake property owners (e.g.) must come at the expense of vegetable growers. It is impossible to make everybody happy all at once.
Optimization is about balancing conflicting objectives. in a quantitative way. How much, and where, should groundwater pumping be limited, so as to minimize the sum of all losses incurred by growers and lake property owners (and all other interests)? Needs a numerical solver - this is what I'm working on.

An optimization model

x groundwater pumping at each well (2000 wells in the region, so $x \in \mathbb{R}^{2000}$)
y stream flows (84 streams, so $y \in \mathbb{R}^{84}$)
Groundwater flow model:

$$y = j(x)$$

MODFLOW (Kraft, Mechenich) 3 seconds on my laptop
net losses:

$$f(x, y) = \sum_w c_w (x_w - x_{w0})^2 + \sum_s d_s (y_s - y_{s0})^2$$

Optimization model:

$$\begin{array}{ll} \min & f(x, y) \\ \text{s.t.} & y = y(x) \end{array}$$

Parameters c, x_0, d, y_0 can be estimated statistically.
Convexity of loss function reflects diminishing returns to scale.

Groundwater optimzation methods: GWM-MODFLOW

Given x
evaluate y
evaluate $a_{sw} = \frac{\partial y_s}{\partial x_w}$ by finite difference
solve a linear programming approximation

$$\begin{aligned} \min \quad & f(x+dx, y+dy) \\ \text{s.t.} \quad & dy = a dx \end{aligned}$$

update $x = x + dx$
repeat until convergence

GWM-MODFLOW pro and con

Dependent on dimension of x. Requires $W + 1$ runs of MODFLOW each doing one evaluation of y 2000 times 3 s = 1:40 h:m

Linear approximation of constraints leads to slow convergence

But because it relies only on solves of MODFLOW the optimization model can be built on top of the groundwater flow model. Very little new modeling is needed.

My goal is to address the drawbacks while still using primarily MODFLOW solves.

Optimization methods: gradient descent

At any point x the negative gradient $-\nabla g$ of g is the direction of steepest desscent of the objective g. So to minimize g, this is the direction we should move in.

given x
evaluate $p = -\nabla g(x)$
update $x = x + tp$ for some suitable choice of $t > 0$

How to choose t? Optimize: $min_t g(x + tp)$

start with $t = 1$
until $g(x + tp) - g(x)$ is sufficiently negative
set $t = 0.5 * t$, repeat

We need to be able to evaluate $g(x)$ and $\nabla g(x)$

Gradient descent for groundwater optimization

In our case this means, evaluate

$$\begin{aligned} y &= y(x) \\ z &= f(x,y) \\ w &= \nabla_x(f(x,y(x))) \\ &= \nabla_x f + \frac{\partial y}{dx}^T \nabla_y f \end{aligned}$$

The hard part is

$$dx^* = \frac{\partial y}{\partial x}^T dy^* \qquad (1)$$

for $dy^* = \nabla_y f$. But we don't need the whole matrix $\frac{\partial y}{\partial x}$. we only need a single matrix-vector (left-) product. The goal is to do this with one, or maybe two, runs of the MODFLOW model or a very closely related model.

Nonlinear solves, linearized solves, and adjoints

x	irrigation	r	recharge	h	head	y	stream flow

Nonlinear solve, for y given x:

$$x \xrightarrow{B} r \rightarrow h \xrightarrow{C} y \tag{2}$$

1 run of MODFLOW

Linearization, for directional derivative of y in the direction dx:

$$dx \xrightarrow{B} dr \xrightarrow{J} dh \xrightarrow{C} dy \tag{3}$$

Finite difference, using 1 or 2 runs of the MODFLOW model

Adjoint solve:

$$dx^* \xleftarrow{B^T} dr^* \xleftarrow{J^T} dh^* \xleftarrow{C^T} dy^* \tag{4}$$

This yields stream-flow-induced prices dx^* on irrigation given prices dy^* on stream flows.

Approximate adjoint solve using MODFLOW

We can compute an approximation of

$$dr^* = J^T dh^*$$

using MODFLOW. The matrix dJ is nonsymmetric because the aquifer is modeled as unconfined, so that the conductance changes as the head changes. If we hold conductance constance and model the aquifer as confined, then the Jacobian $\hat{J}$ for the resulting system is a symmetric approximation of the true Jacobian J. So

$$dr^* = J^T dh^* \approx \hat{J}^T dh^* = \hat{J} dh^*$$

We can compute Jdh^* by 1 solve of the confined aquifer model. Thus we obtain an approximate gradient for the optimization problem.

Progress so far

- Matlab routines calling MODFLOW to compute $y(x)$ and the approximate gradient.
- There is sometimes a delicate balance among the objective scale, the step length scale, and the MODFLOW termination criteria.
- Acceleration of steepest descent by conjugate gradient.

Thanks

- Thanks to George Kraft and David Mechenich for letting me work with their model.
- Thanks to all of you for listening.